HIGHLY UNSTEADY HEAT AND MASS TRANSFER IN A REGION WITH MOVING BOUNDARIES WHEN THE KINETIC EQUATIONS ARE UNKNOWN

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The traveling-wave method is used to obtain a solution of the system of heat- and mass-transfer equations, which is analyzed at small times.

In analyzing heat and moisture transfer occurring rapidly in conditions of external shock perturbation, account must be taken of the finite rate of heat and mass transfer. In many cases, it is difficult to study such processes in which phase transition occurs, because of the lack of kinetic equations describing the mechanism of new-phase formation close to the phase boundary at small times.

Below, the problem of highly intense heat and mass transfer is solved in the case when the values of the potentials and flux densities at the motionless thermal boundary are known.

The formulation of the problem is as follows. On the basis of the system of equations inside the phase [1]

$$\tau_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a_T \frac{\partial^2 T}{\partial x^2} - \alpha \frac{\partial u}{\partial t}, \tag{1}$$

$$\tau_2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = a_m \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^2 T}{\partial x^2}, \tag{2}$$

the values of the potentials and gradients at the motionless boundary

$$T(0,\ t) = A_1(t);\quad \frac{\partial T}{\partial x}(0,\ t) = A_2(t),\ t \geqslant 0, \tag{3}$$

$$u(0,\ t) = B_1(t);\quad \frac{\partial u}{\partial x}(0,\ t) = B_2(t),\ t \geqslant 0, \tag{4}$$

the heat and mass balance conditions at the motionless boundary [2]

$$\frac{\partial T}{\partial x}(R(t),\ t) = \gamma(\tau_1 R'' + R') + q_T, \tag{5}$$

$$\frac{\partial u}{\partial x}(R(t),\ t) = \theta(\tau_2 R'' + R') + q_m \tag{6}$$

and the conditions on the initial position of the interphase boundary

$$R(0) = R_0;\ R'(0) = R_0',$$

it is required to calculate the fields T(x; t) and u(x; t) and the law of motion of the mobile boundary x = R(t).

The system in Eqs. (1) and (2) is transformed so that it may be written in matrix form. Equation (2) is multiplied by a fixed number $g_1 \neq 0$ and added to Eq. (1). Analogously, Eq. (2) is multiplied by $g_2 \neq g_1$ ($g_2 \neq 0$) and added to Eq. (1), to obtain the second equation of the system. The numbers g_1 and g_2 are chosen so that the coefficients of

$$\frac{\partial^2 T}{\partial x^2};\ \frac{\partial^2 u}{\partial x^2};\ \frac{\partial^2 T}{\partial t^2};\ \frac{\partial^2 u}{\partial t^2};\ \frac{\partial T}{\partial t};\ \frac{\partial u}{\partial t}$$

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are nonzero in each equation obtained. For example, taking $g_1 = 1$, $g_2 = -1$, the system obtained is

$$\left(\tau_1 \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} - \left(a_T + \beta g_i \frac{\partial^2}{\partial x^2}\right)\right) T(x, t) + \\ + \left(\tau_2 g_i \frac{\partial^2}{\partial t^2} + (g_i + \alpha) \frac{\partial}{\partial t} - g_i a_m \frac{\partial^2}{\partial x^2}\right) u(x, t) = 0 \ (i = 1, 2). \tag{7}$$

Introducing the notation

$$a_{11} = a_{21} = \tau_1;\ a_{12} = g_1\tau_2;\ a_{22} = g_2\tau_2;\ b_{11} = b_{21} = 1; \\ b_{12} = (g_1 + \alpha);\ b_{22} = g_2 + \alpha; -c_{11} = a_T + \beta_1 g_1; \\ -c_{21} = a_T + \beta g_2;\ c_{12} = -g_1 a_m;\ c_{22} = g_2 a_m. \tag{8}$$

Eq. (7) may be written in matrix form

$$AN = Q, \tag{9}$$

where

$$A = \begin{pmatrix} a_{11}\frac{\partial^2}{\partial t^2} + b_{11}\frac{\partial}{\partial t} + c_{11}\frac{\partial^2}{\partial x^2} & a_{12}\frac{\partial^2}{\partial t^2} + b_{12}\frac{\partial}{\partial t} + c_{12}\frac{\partial^2}{\partial x^2} \\ a_{21}\frac{\partial^2}{\partial t^2} + b_{21}\frac{\partial}{\partial t} + c_{21}\frac{\partial^2}{\partial x^2} & a_{22}\frac{\partial^2}{\partial t^2} + b_{22}\frac{\partial}{\partial t} + c_{22}\frac{\partial^2}{\partial x^2} \end{pmatrix} \tag{10}$$

$$N = \begin{pmatrix} T(x, t) \\ u(x, t) \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

Note that Eq. (9) is more general in form than the initial system. The solution of Eq. (9) is sought in the form of a series

$$N = \sum_{n=0}^{\infty} F_n X_n, \tag{12}$$

where

$$\begin{Bmatrix} F_n = \begin{pmatrix} r_{n11}(t) & r_{n12}(t) \\ r_{n21}(t) & r_{n22}(t) \end{pmatrix} \\ X_n = \begin{pmatrix} \dfrac{x^{2n}}{(2n)!} \\ \dfrac{x^{2n+1}}{(2n+1)!} \end{pmatrix} \end{Bmatrix} \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (9) and comparing the coefficients of the same powers of x, a recurrence relation for the unknown functions is obtained

$$F_{n+1} = DF_n, \tag{14}$$

where the elements d_{ij} of the matrix D, with the structure $d_{ij} = A_{ij}\partial^2/\partial t^2 + B_{ij}\partial/\partial t$, take the specific form

$$d_{11} = \left(\begin{vmatrix} c_{11} & a_{11} \\ c_{22} & a_{21} \end{vmatrix} \frac{\partial^2}{\partial t^2} + \begin{vmatrix} c_{12} & b_{11} \\ c_{22} & b_{21} \end{vmatrix} \frac{\partial}{\partial t}\right) : \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \\ d_{12} = \left(\begin{vmatrix} c_{12} & a_{12} \\ c_{22} & a_{22} \end{vmatrix} \frac{\partial^2}{\partial t^2} + \begin{vmatrix} c_{12} & b_{12} \\ c_{22} & b_{22} \end{vmatrix} \frac{\partial}{\partial t}\right) : \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \\ d_{21} = \left(\begin{vmatrix} c_{21} & a_{21} \\ c_{11} & a_{11} \end{vmatrix} \frac{\partial^2}{\partial t^2} + \begin{vmatrix} c_{21} & b_{21} \\ c_{11} & b_{11} \end{vmatrix} \frac{\partial}{\partial t}\right) : \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \\ d_{22} = \left(\begin{vmatrix} c_{21} & a_{22} \\ c_{11} & a_{12} \end{vmatrix} \frac{\partial^2}{\partial t^2} + \begin{vmatrix} c_{21} & b_{22} \\ c_{11} & b_{12} \end{vmatrix} \frac{\partial}{\partial t}\right) : \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \tag{15}$$

Hence,

$$F_n = D^n F_0, \tag{16}$$

and F_0 is expressed in terms of $T(0, t)$, $u(0, t)$, $T_x'(0, t)$, $u_x'(0, t)$ as follows

$$F_0 = \begin{pmatrix} T(0, t) & T_x'(0, t) \\ u(0, t) & u_x'(0, t) \end{pmatrix} \tag{17}$$

Thus, the solution of Eq. (9) takes the form

$$\begin{pmatrix} T(x, t) \\ u(x, t) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}^n \begin{pmatrix} T(0, t) & T_x'(0, t) \\ u(0, t) & u_x'(0, t) \end{pmatrix} \begin{pmatrix} \dfrac{x^{2n}}{(2n)!} \\ \dfrac{x^{2n+1}}{(2n+1)!} \end{pmatrix} \tag{18}$$

Since the matrix D is formed by differentiation operators, it is necessary to specify the conditions under which the series in Eq. (18) converges to its sum. Attention is confined here to the case when the functions $T(0, t)$, $T_x'(0, t)$, $u(0, t)$, and $u_x'(0, t)$ are approximated by polynomials or finite sums of sines, cosines, exponentials.

In these cases, it is obvious that the series in Eq. (18) may be turned into a sum.

One of the balance Eqs. (5) and (6) may be used to determine $R(t)$.

One important particular case of the solution obtained corresponding qualitatively to the classical solution of th Stefan problem, which is often used in practice, is considered here [3, 4]. As is known, the Stefen solutions of separate thermal and water-conduction problems are expressed in terms of the error function erf(z)

$$\begin{aligned} T(x, t) &= T_0 + p_0 \operatorname{erf}(x/\sqrt{4a_1 t}), \\ u(x, t) &= u_0 + f_0 \operatorname{erf}(x/\sqrt{4a_2 t}), \\ R(t) &= \tilde{\alpha}\sqrt{t}. \end{aligned} \tag{19}$$

They correspond to the boundary function

$$\begin{aligned} &T(0, t) = T_0 = \text{const}, \ u(0, t) = u_0 = \text{const}, \\ &T_x'(0, t) = p_0/\sqrt{\pi a_1 t}, \ u_x'(0, t) = f_0/\sqrt{\pi a_2 t}. \end{aligned} \tag{20}$$

This is the solution of the inadequately described process at small times. In fact, $T_x'(0, t) \to \infty$ and $u_x'(0, t) \to \infty$ as $t \to 0$; the front velocity behaves in the same way: $R'(t) \to \infty$ as $t \to 0$.

With constant $T(0, t) = T_0 = \text{const}$, $u(0, t) = u_0 = \text{const}$. Let $T_x'(0, t) = q_{01} \exp(-\varphi t)$, $u_x'(0, t) = q_{02} \exp(-\varphi t)$. The values of the gradients begin to change from the value of greatest modulus and, decreasing smoothly, tends to zero at $t \to \infty$, i.e., behaves qualitatively in the same way as the solution in Eq. (19).

Substituting the boundary values into Eq. (18) leads to the relations

$$\begin{aligned} T(x, t) = T_0 + \exp(-\varphi t)\,(q_{01} \sin((\varphi B_{11} - \varphi^2 A_{11})^{1/2} x) &: \\ : (\varphi B_{11} - \varphi^2 A_{11})^{1/2} + q_{02} \sin((\varphi B_{12} - \varphi^2 A_{12})^{1/2} x) &: \\ : (\varphi B_{12} - \varphi^2 A_{12})^{1/2}), & \end{aligned} \tag{21}$$

$$\begin{aligned} u(x, t) = u_0 + \exp(-\varphi t)\,(q_{01} \sin((\varphi B_{21} - \varphi^2 A_{21})^{1/2} x) &: \\ : (\varphi B_{21} - \varphi^2 A_{21})^{1/2} + q_{02} \sin((\varphi B_{22} - \varphi^2 A_{22})^{1/2} x) &: \\ : (\varphi B_{22} - \varphi^2 A_{22})^{1/2}). & \end{aligned} \tag{22}$$

Neglecting the relaxation time on the right-hand side of Eq. (5), i.e., taking the heat finiteness of the heat-propagation velocity into account only inside the phase, the following equation is obtained for determining $R(t)$

$$\begin{aligned} &\exp(-\varphi t)\,(q_{01} \cos((\varphi B_{11} - \varphi^2 A_{11})^{1/2} R(t)) + \\ &+ q_{02} \cos((\varphi B_{12} - \varphi^2 A_{12})^{1/2} R(t))) = \gamma R'. \end{aligned} \tag{23}$$

To simplify the subsequent computations, attention is confined to the case when φ satisfies the relation

$$\varphi B_{11} - \varphi^2 A_{11} = \varphi B_{12} - \varphi^2 A_{12} = \omega. \tag{24}$$

Then the following expression is obtained for R(t) when R(0) = 0

$$R(t) = \frac{2}{\omega}\left(\operatorname{arctg}\left(\exp\left(\frac{(q_{01}+q_{02})\,\omega}{\varphi\gamma}\left(1-\exp(-\varphi t)\right)\right)\right)-\frac{\pi}{4}\right). \tag{25}$$

In contrast to the classical solution $R(t) = \tilde{\alpha}\sqrt{t}$, for which $R(\infty) = \infty$, the function R(t) determined by Eq. (25) is finite at infinity

$$R(\infty) = \frac{2}{\omega}\left(\operatorname{arctg}\left(\exp\left(\frac{(q_{01}+q_{02})\,\omega}{\varphi\gamma}\right)\right)-\frac{\pi}{4}\right). \tag{26}$$

The values of the velocities in the solutions of Eqs. (19) and (24) as $t \to \infty$ are qualitatively identical; they both tend to zero; initially, however, they are significantly different. In the solution in Eq. (19), $R'(t) \sim 1/\sqrt{t} \to \infty$ as $t \to 0$. The solution in Eq. (25) corresponds to a finite value of R'(0)

$$R'(0) = \gamma(q_{01} + q_{02}).$$

In addition, in contrast to the classical case, when $\tau_i \neq 0$, the problem in Eqs. (1)-(6) leads to stable solutions with respect to small changes in the input data.

Thus, as noted in [2, 5, 6], the parameters τ_i play the role of "natural regularization parameters." The method of obtaining the required error as a function of the errors of the initial boundary functions was described in [7].

NOTATION

τ_1, τ_2, relaxation times of the transfer processes; T, temperature; u, moisture content; a_t, thermal diffusivity; a_m, diffusion coefficient of moisture; $\beta = a_m\delta$; δ, thermogradient coefficient; α, quantity proportional to the ratio of the specific heat of phase transition and the specific heat; γ, ratio of the specific heat of phase transition to the product of the thermal diffusivity and the specific heat; R(t), law of motion of phase-transition front.

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